Lattice reduction based MIMO detection and its application to multiuser systems

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Outline

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2. MIMO Detection
3. Lattice Reduction based Detection
4. LR based List Detection
5. LR based Detection for MIMO MAC
6. Concluding Remarks
1. Introduction

- The use of multiple antennas has been studied to increase spectral efficiency.
- The resulting channel is called multiple input multiple output (MIMO) channel.

\[
\begin{align*}
N_T \times 1 & \quad N_R \times 1 \\
N_T \text{ transmit antennas} & \quad N_R \text{ receive antennas}
\end{align*}
\]
Capacity of MIMO Channel

- Capacity formula: \( C = \min(N_T, N_R) \log_2(1 + SNR) \)

In wireless communications, power is not an effective means to increase the capacity due to propagation loss (received power \( \propto \) tx power \( \times d^{-r}, r = 3 \) or \( 4 \)). Hence, MIMO provides a way to improve capacity without increasing power (or SNR).
How to Achieve MIMO Capacity

- Capacity Achieving Codes
- Higher Order Modulation
- ML or Near ML Decoder (Detector)
An example

- $4 \times 4$
- Target capacity = 12 bps/Hz/channel use

Code rate = $\frac{1}{2}$

Modulation order becomes

$$M = 2^{\frac{12 \times 2}{4}} = 64$$
2. MIMO Detection

- In MIMO systems, signal detection for *higher order modulation* is necessary.
- There is a trade-off between complexity and performance.

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<td>High (growing exponentially)</td>
<td>Sphere decoding (complexity depends on channel matrices)</td>
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<td>Linear (MMSE/ZF)</td>
<td>No good</td>
<td>Low</td>
<td>LR based algorithms (complexity depends on channel matrices)</td>
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<td>SIC (MMSE/ZF)</td>
<td>No good</td>
<td>Low</td>
<td>Partial MAP/LR based algorithms (complexity depends on channel matrices)</td>
</tr>
</tbody>
</table>
ML Detection

- Received signal vector:
  \[ r = Hs + n \]

- The ML detection
  \[ s_{ML} = \arg \max_{s \in S^{NT}} \exp \left( -\frac{1}{N_0} \| r - Hs \|^2 \right) \]
  \[ = \arg \min_{s \in S^{NT}} \| r - Hs \|^2 \]

  Due to the number of candidate vectors for searching, its complexity grows exponentially.

- Sphere decoding (SD) can reduce the complexity significantly, but the complexity depends on \( H \).
SIC Detection

\[
\begin{bmatrix}
  r_1 \\
  r_2
\end{bmatrix} = \begin{bmatrix}
  h_{11} & h_{12} \\
  h_{14} & h_{12}
\end{bmatrix} \begin{bmatrix}
  s_1 \\
  s_2
\end{bmatrix} + \begin{bmatrix}
  n_1 \\
  n_2
\end{bmatrix}
\]

\[
= QR
\]

\[
\Leftrightarrow Q^H r = \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = \begin{bmatrix}
  a_1 & \phi \\
  0 & 0
\end{bmatrix} \begin{bmatrix}
  s_1 \\
  s_2
\end{bmatrix} + \begin{bmatrix}
  w_1 \\
  w_2
\end{bmatrix}
\]

QR factorization:

\[
H = QR
\]

- Low complexity, but the performance suffers from error propagation.
- However, there are several approaches to mitigate error propagation:
  - Power and rate allocations
  - Selective cancellation (Partial MAP principle)
  - List detection
Partial MAP Principle (to reduce complexity of ML detector)

• Let us consider the following received signal:

\[ x_1 = a_1 s_1 + \phi s_2 + n_1 \]

Assume that the a priori probability of \( s_2 \) is known, but not \( s_1 \). Then, the partial MAP detection is now given by

\[
\begin{align*}
(\hat{s}_1, \hat{s}_2) &= \arg \max_{s_1, s_2} P(s_1, s_2 | x_1) \\
&= \arg \max_{s_1, s_2} f(x_1 | s_1, s_2) P(s_1) P(s_2) \\
&= \arg \max_{s_1, s_2} f(x_1 | s_1, s_2) P(s_2)
\end{align*}
\]

Joint Detection
Optimality of SIC based on partial MAP

**Theorem:** Let \( \bar{s}_2 = \max_{s_2} P(s_2) \)

If \( \frac{P(\bar{s}_2)}{\max_{s_2 \neq \bar{s}_2} P(s_2)} > V \), then

\[
\hat{s}_2 = \bar{s}_2 = \arg \max_{s_2} P(s_2)
\]

\[
\hat{s}_1 = \arg \max_{s_1} f(x_1 | s_1, s_2 = \bar{s}_2)
\]

\[
= \arg \min_{s_1} \| x_1 - \phi \bar{s}_2 - a_1 s_1 \|^2
\]

This is the cancellation based detection!
SIC Detection with Partial MAP Principle

- **2 × 2 case**: After QR factorization of $H = QR$

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

Then, perform the partial MAP detection with the a posteriori probability of $s_2$ as the a priori probability.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 & \phi \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Perform the ML detection and find the metrics and then compute the a posteriori probability of $s_2$. 
Application to MIMO detection

- According to the partial MAP detection theorem:
  - Cancellation based on hard-decision is optimal when the previous symbols are reliably determined
  - Otherwise, soft-decision based cancellation can be used.

An example:
3. Lattice Reduction based Detection

- Lattice reduction (LR) can be used to decode lattice coded signals.
- This approach has been applied to MIMO detection by Yao in 2002.
- Wubben et. al extended this LR based detection further in 2004.
- It has been reported that the LR based MIMO detection can achieve a full receive diversity.
  - It means that the BER performance will be the same as the ML performance except for an offset
Distortion due to fading channel

\[ r = Hs + n \]

**s**: Signal vector with orthogonal basis

\( s_1, s_2 \): 4-PAM

Tilted due to channel matrix:

Basis vectors are **not** orthogonal
Lattice (Basis) Reduction

\[ H = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, s_k \in \{0,1,2,3\} \]

Find a new set of basis vectors that are (nearly) orthogonal over lattice
Linear Detection in the LR Domain

- LR over basis vectors of channel matrix

\[ H = AU \]

Integer matrix

- LR based detection

\[ r = AU_c s + n = Ac + n \]

Nearly orthogonal

\[ \hat{c} = A^+ r = c + A^+ n \]

\[ \hat{s} = U^{-1} [\hat{c}] \]

Zero-forcing or MMSE

Detecting \( c \) and transform into the original domain
SIC Detection in the LR Domain

• QR factorization with $A$:

$$H = AU = QRU$$

• SIC-LR based detection

$$Q^H r = Q^H (QRUs + n)$$

$$= R\left(\begin{bmatrix} Us \\ c \end{bmatrix}\right) + Q^H n$$

$$\hat{c} = \text{SIC\_Detection}(Q^H r, R)$$

$$\hat{s} = U^{-1}[\hat{c}]$$
Simulation Results

BER vs. $E_b/N_0$ (dB) with a 1 dB gap.
Issues with LR based detection

- LR requires a polynomial time complexity
  - The complexity depends on channel matrices
  - The complexity can be prohibitively high when the size of channel matrices is large
- LR based detectors produce hard-decision

Approaches to solve the above problems:

- Complexity reduction ⇒ Reduce the size of channel matrices using the notion of SIC
- Soft-decision ⇒ use list to produce soft-decision
4. LR based List Detection

• LR algorithms require a polynomial time complexity.

• Thus, for a large number of transmit antennas, the complexity can be still high.

• In addition, there is a performance gap from the ML detector and this gap increases with the size of constellation.
Breaking a MIMO channel into pieces

- QR factorization:

\[
\begin{align*}
r & = Hs + n \\
& = QRs + n \\
Q^H r & = Rs + Q^H n
\end{align*}
\]

\[
\begin{bmatrix}
R_1 & \Phi \\
0 & R_2
\end{bmatrix}
\begin{bmatrix}
s_1 \\
s_2
\end{bmatrix}
+ \begin{bmatrix}
n_1 \\
n_2
\end{bmatrix}
\]

Two sub-detection
Problem based on
Partial-MAP principle

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
R_1 s_1 + \Phi s_2 + n_1 \\
R_2 s_2 + n_2
\end{bmatrix}
\]
List Detection of $s_2$

- List of candidates for $s_2$

\[
\hat{s}_2 = \text{Detection}(x_2)
\]

\[
\{\hat{s}_2^{(1)}, \hat{s}_2^{(2)}, K, \hat{s}_2^{(Q)}\} = \text{List Detection}(x_2, Q)
\]

- The detection of $s_2$ cannot enjoy a full diversity due to nulling. Thus, in general, it is not reliable.

- However, by having a list, the probability of error can be effectively reduced as error propagation can be mitigated.
SIC Detection of $s_1$

- Detection after SIC

$$\hat{s}_1^{(q)} = \text{Detection} \left( x_1 - \Phi \hat{s}_2^{(q)} \right), \quad q = 1, 2, \ldots, Q$$

- Choose the best or keep the list

$$\min_q \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} R_1 & \Phi \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} \hat{s}_1^{(q)} \\ \hat{s}_2^{(q)} \end{bmatrix} \right\|^2$$
LR based Detection with List

- Each sub-detection can be carried out using the LR based detection.
- Since the dimension is reduced, the complexity becomes lower.
- The list construction can be done in the LR domain with low complexity.
Error Probability with List

- To build a list, two distance measures can be used:
  - Mahalanobis (optimal)
  - Euclidean distance (suboptimal)

- Euclidean distance becomes good approximation (LR domain is nearly orthogonal)
Performance and Complexity

- The longer list length, the better performance.
- However, the complexity increases.
- Therefore, the list length, $Q$, enjoys the trade-off between the performance and complexity.
- Complexity: less than 3 times of MMSE detector

<table>
<thead>
<tr>
<th>Decoder</th>
<th>Complex multiplications</th>
<th>4-QAM</th>
<th>16-QAM</th>
<th>64-QAM</th>
<th>$Q = 8$</th>
<th>$Q = 12$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MMSE</td>
<td>$2(K + 1)K^2$</td>
<td>160</td>
<td>160</td>
<td>160</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>ML</td>
<td>$K^2</td>
<td>S</td>
<td>^K$</td>
<td>4096</td>
<td>1048576</td>
<td>268435456</td>
</tr>
<tr>
<td>LR+List</td>
<td>$K^2(5K + 12)/8 + QK^2 + 2C\mathcal{P}<em>{K/2} + \sum</em>{k=1}^{K} 2(K-k+1)^2$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>262</td>
<td>326</td>
</tr>
</tbody>
</table>

$Q=16$

390
Simulation Results (1)

- $4 \times 4$
- 4-QAM
Simulation Results (2)

- $4 \times 4$
- 16-QAM
Simulation Results (3)

- 4 × 4
- 64-QAM
5. LR based Detection for MIMO MAC

- Uplink in a cellular system: only one user can access the channel
- Suitable for MIMO-OFDM

\[ r = H_k s_k + n \]

If user \( k \) is chosen
User selection – multiuser diversity (1)

- Maximum capacity based user selection

\[ k^* = \arg \max_{1 \leq k \leq K} \log \det \left( \mathbf{I} + \rho_k \mathbf{H}_k \mathbf{H}_k^H \right) \]

- Scaling law

\[ \log \det \left( \mathbf{I} + \rho_{k^*} \mathbf{H}_{k^*} \mathbf{H}_{k^*}^H \right) \sim Q \log \log K \]

- This capacity based user selection is ideal to maximize the throughput. But, channel adaptive and capacity achieving codes and ML decoding have to be used.
User selection – multiuser diversity (2)

- If practical and non-ideal codes are used, the user selection criterion should be changed.
- Error probability can be a good criterion

\[ k^* = \arg \min_{1 \leq k \leq K} P_{err}(\rho_k, H_k) \]

- Error probability depends on detector.
  - ML detector
  - **LR based detector** (we only show the selection criterion for this detector)
  - etc
User selection: LR based detection employed

- If a low complexity detector is employed, we may need to use different user selection criterion.

- When SIC-LR based detector is employed, the performance depends on the diagonal elements of $R$.

$$Q^H r = Q^H (QRUs + n)$$

$$= R \left( \begin{array}{c} Us \\ \vdots \end{array} \right) + Q^H n$$

$$\hat{c} = \text{SIC\_Detection}(Q^H r, R)$$

$$\hat{s} = U^{-1}[\hat{c}]$$
User selection criterion from error probability

\[
\Pr(\text{error}) \approx \sum_{q=1}^{Q} \exp \left( - \frac{\left| r_{q,q} \right|^2}{4N_0} \right)
\]

\[
\leq Q \exp \left( - \min_{q} \frac{\left| r_{q,q} \right|^2}{4N_0} \right)
\]

\[
\Rightarrow \quad k^* = \arg \max_{1 \leq k \leq K} \min_{1 \leq q \leq N_T} \frac{\left| \left[ R_{(k)} \right]_{q,q} \right|^2}{4N_0}
\]

- We can derive user selection criteria for other detectors as well using error probability.
ML detector

16-QAM

\( K = 10 \)

\( N_T = N_R = 4 \)

Criterion has to be chosen to fully exploit

- multiuser diversity
- receive diversity
LR based detectors

16-QAM

$K = 10$

$N_T = N_R = 4$
6. Concluding Remarks

• MIMO detection often requires a prohibitively high complexity to achieve a good performance.
• LR based MIMO detection and SIC based detection play a key role in reducing complexity without a significant performance loss.
• A well-designed LR based MIMO detector has a good performance which is within 1 dB from ML performance with less than 3 times complexity of linear MMSE detector (for 16-QAM and 64-QAM).
• These new low complexity MIMO detection methods are also important in deriving user selection criteria for multiuser MIMO systems.