Frequency Domain Equalization (FDE)

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- FDE with Cyclic Prefixes (CP)
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- Simulation Results
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Consider a matrix system

\[ r = Hx + \eta \]

where
- \( r \) is the observation vector,
- \( x = \{x_j\} \) are binary inputs,
- \( H = [h_1, h_2, \ldots, h_j, \ldots,] \) is a constant matrix and
- \( \eta \) is a vector of AWGN samples.
The Overall Systems Model

\[ r = Hx + \eta \]
Different Approaches to the ESE

Good performance and relatively low-cost

Decoder (APP) → Signal Estimator (MAP) → Good performance but prohibitively high cost

Decoder (APP) → Signal Estimator (LMMSE) → Good performance and relatively low-cost
How can we further reduce complexity?

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The JG Estimator

The JG estimator is given by

$$\lambda_j = \ln \frac{p(r \mid x_j = +1)}{p(r \mid x_j = -1)} = 2 \frac{h_j^T R^{-1}(r - HE(x)) + h_j^T R^{-1} h_j E(x_j)}{1 - \nu_j h_j^T R^{-1} h_j}$$

Let $U = (HR^{-1}H)_{\text{diag}}$. We can rewrite it in a compact vector form as

$$\lambda = 2(I - VU)^{-1} [H^T R^{-1} (r - HE(x)) - UE(x)]$$

For detail, see
The Computational Problem

The JG estimator:

\[
\lambda = 2(I - VU)^{-1}\left[H^T R^{-1}(r - H\mathbb{E}(x)) - U\mathbb{E}(x)\right]
\]

\[
R = HVH^T + \sigma^2 I
\]

\[
U = (HR^{-1}H)_{\text{diag}}
\]

Consider the evaluation of \(\lambda\). The key problems is related to \(R\) and \(H\). The technique described below is to transform \(H\) into a circulant matrix, so that all the operations can be performed efficiently.

\[
H = \begin{bmatrix}
h_0 & h_2 & h_1 \\
h_1 & h_0 & h_2 \\
h_2 & h_1 & h_0 \\
h_2 & h_1 & h_0 \\
h_2 & h_1 & h_0 \\
h_2 & h_1 & h_0 \\
\end{bmatrix}
\]
The CP Technique

- Insert Cyclic Prefix (CP) before transmission.
- Remove CP from the received signal.
- As the result, it converts linear convolution to cyclic convolution. It also converts $H$ into a circulant matrix

\[
H = \begin{bmatrix}
  h_0 & h_1 & h_2 & h_1 \\
  h_1 & h_0 & h_2 & h_2 \\
  h_2 & h_1 & h_0 & h_2 \\
  h_2 & h_1 & h_0 & h_2 \\
  h_2 & h_1 & h_0 & h_2 \\
  h_2 & h_1 & h_0 & h_2 \\
  h_2 & h_1 & h_0 & h_2 \\
  h_2 & h_1 & h_0 & h_2 \\
\end{bmatrix}
\]
The Power Efficiency Problem

$H$ is circulant only after removing CP. (In other words, $H$ is part of a non-circulant matrix.) This means that the energy in CP is not used in detection, which incurs an overhead.

$H = \begin{bmatrix}
  h_0 & h_2 & h_1 \\
  h_1 & h_0 & h_2 \\
  h_2 & h_1 & h_0 \\
  h_2 & h_1 & h_0 \\
  h_2 & h_1 & h_0 
\end{bmatrix}$

Transmitted signal: $\text{CP} \quad x$

Received signal: $\text{CP} \quad r = Hx + \eta$
The Computational Problem

\[ \lambda = 2(I - VU)^{-1}[H^T R^{-1}(r - HE(x)) - UE(x)] \]
\[ R = HVH^H + \sigma^2 I \]
\[ U = (HR^{-1}H)_{\text{diag}} \]

Let us consider the evaluation of \( \lambda \). The key problems is related to \( R \) and \( H \). Our approach below is to transform \( H \) into a circulant matrix, so that all the operations can be performed efficiently.

\[
H = \begin{bmatrix}
  h_0 & h_2 & h_1 \\
  h_1 & h_0 & h_2 \\
  h_2 & h_1 & h_0 \\
  h_2 & h_1 & h_0 \\
  h_2 & h_1 & h_0 \\
  h_2 & h_1 & h_0
\end{bmatrix}
\]
Efficient Approach to the JG Estimator

The properties for a circulant matrix $H$

\[
G = FHF^H = \begin{bmatrix}
J^{1/2}g_0 \\
& J^{1/2}g_1 \\
& & \ddots \\
& & & J^{1/2}g_{J-1}
\end{bmatrix}
\]

\[
H = F^HGF
\]

where $F$ is the normalized DFT matrix, and $\{g_i\}$ is the DFT of $\{h_i\}$. The above decomposition indicates that the operation related to $H$ can be performed efficiently using FFT.
Efficient Implementation using FFT

The general JG estimator

\[ \lambda = 2(I - VU)^{-1}[H^T R^{-1}(r - HE(x)) - UE(x)] \]
\[ R = HVH^H + \sigma^2 I \]
\[ U = (HR^{-1}H)_{\text{diag}} \]

The circulant JG estimator

\[ \lambda = 2(1-vu)^{-1}(F^H G^H(\nu GG^H + \sigma^2 I)^{-1}(Fr - GF E(x)) + uE(x)) \]
\[ u = \sum_{j=0}^{J-1}|g_j|^2 (\nu J|g_j|^2 + \sigma^2)^{-1} \]

This is because

\[ H = F^H GF \]
\[ R^{-1} = (HVH^H + \sigma^2 I)^{-1} = F^H(\nu GG^H + \sigma^2 I)^{-1} \]

The complexity involving \( F \) is \( O(\log J) \) per entry using FFT.
**Frequency Domain Equalization (FDE)**

The circulant JG estimator

\[ \lambda = 2(1-\nu u)^{-1}(F^H G^H \left( \nu G G^H + \sigma^2 I \right)^{-1} (Fr - GFE(x)) + uE(x)) \]

Observations:
- \( Fr \) is the Fourier transform of \( r \).
- \( G^H(\nu GG^H + \sigma^2 I)^{-1} \) is FDE.
- Multiple by \( F \) again transforms the results back to the time domain.

Thus this is a different form of FDE.

For detail, see

MIMO-ISI Block Circulant Systems

Here each $H^{(m,n)}$ is a circulant matrix representing the ISI channel from antenna $m$ to antenna $n$. Then

$$r = \begin{bmatrix} r^{(1)} \\ r^{(2)} \end{bmatrix}, \quad Hx = \begin{bmatrix} H^{(1,1)} & H^{(1,2)} \\ H^{(2,1)} & H^{(2,2)} \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix},$$

Each $G^{(m,n)}$ is diagonal. $F$ is an DFT matrix.
A multi-user-MIMO-ISI system can be decomposed similarly.

\[
H = \begin{bmatrix}
H^{(1,1,1)} & H^{(1,2,1)} & H^{(1,1,2)} & H^{(1,2,2)} \\
H^{(2,1,1)} & H^{(2,2,1)} & H^{(2,1,2)} & H^{(2,2,2)}
\end{bmatrix}
\]

for user 1

for user 2

\[
\begin{pmatrix}
F \\
F
\end{pmatrix}
\begin{pmatrix}
G^{(1,1,1)} & G^{(1,2,1)} & G^{(1,1,2)} & G^{(1,2,2)} \\
G^{(2,1,1)} & G^{(2,2,1)} & G^{(2,1,2)} & G^{(2,2,2)}
\end{pmatrix}
\begin{pmatrix}
F \\
F \\
F
\end{pmatrix}
\]
A Not for Multi-user Systems

More users normally do not lead to higher complexity. This is because complexity is dominated by computing the inverse of $R = HVH^H + \sigma^2 I$. Here is an illustration.

$$R = H \begin{bmatrix} V & H^H \end{bmatrix} + \sigma^2 I$$

where $V$ is diagonal. We can see that more users will not increase the size of $R$. Hence it will not increases complexity too much.

$$H = \begin{bmatrix} H^{(1,1,1)} & H^{(1,2,1)} & H^{(1,1,2)} & H^{(1,2,2)} \\ H^{(2,1,1)} & H^{(2,2,1)} & H^{(2,1,2)} & H^{(2,2,2)} \end{bmatrix}$$

for user 1

for user 2
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The drawbacks of the CP technique
- CP reduces power efficiency.

The ZP technique
- Replace CP with zero-padding (ZP).
- High power efficiency

For detail, see
We add zero padding between every two consecutive blocks. This ZP technique is equivalent to CP.
The Zero Padding (ZP) Technique

We can achieve **100% efficiency** of the received energy with ZP! Therefore ZP is more energy efficient than CP.
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Comparison of FDE-CP and FDE-ZP

System Settings:
Rate-1/2 convolution code
Generator=(23, 35)$_8$
Length -16 repetition code
QPSK modulation
Number of users = 32
Sum rate = 2
Block size $N = 64$.
Channel memory length $L = 16$
$L = (1/4)N$. 

![Comparison of FDE-CP and FDE-ZP](image-url)
FDE-IDMA and OFDM-IDMA

**System Settings:**
Rate-1/2 convolution code
Generator=$(23, 35)_8$
Length -16 repetition code
QPSK modulation
Number of users = 32
Sum rate = 2
Block size $N = 64$
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$L = (1/4)N$. 

![Graph showing BER vs. Eb/No for different systems]
Conclusions

- The circulant matrix method provides a fast implementation technique for the JG estimator.
- This method is in principle equivalent to FDE. However, the new approach is more pragmatic, more concise, more flexible and provide more insights.
- This technique can be easily extended to multi-user-MIMO-ISI environments.
- FDE-ZP can save energy compared with FDE-CP.
Thanks!